Solution 10

Supplementary Problems

1. Verify Green's theorem when the region D is the rectangle $[0, a] \times [0, b]$.

   Solution. The boundary of the rectangle consists of four curves: $C_1, x \mapsto (x, 0), x \in [0, a]; C_2, y \mapsto (a, y), y \in [0, b]; C_3, x \mapsto (x, b), x \in [0, a]; C_4, y \mapsto (0, y), y \in [0, b]$ and $C = C_1 + C_2 - C_3 - C_4$. We have

$$\int_{C_1} M dx + N dy = \int_0^a M(x, 0) dx,$$

$$\int_{C_2} M dx + N dy = \int_0^b N(a, y)\, dy \ ,$$

$$\int_{C_3} M dx + N dy = \int_0^a M(x, b)\, dx \ ,$$

$$\int_{C_4} M dx + N dy = \int_0^b N(0, y)\, dy \ .$$

   It follows that

$$\begin{aligned} \int_C M dx + N dy &= \left(\int_{C_1} + \int_{C_2} - \int_{C_3} - \int_{C_4} \right) M dx + N dy \\ &= \int_0^a M(x, 0) dx + \int_0^b N(a, y)\, dy - \int_0^a M(x, b)\, dx - \int_0^b N(0, y)\, dy \ . \end{aligned}$$

   On the other hand,

$$\begin{aligned} \iint_D (N_x - M_y)\, dA &= \iint_D N_x dA - \iint_D M_y dA \\ &= \int_0^b \int_0^a N_x dx dy - \int_0^a \int_0^b M_y dy dx \\ &= \int_0^b N(a, y) dy - \int_0^b N(0, y)\, dy - \int_0^a M(x, b) dx + \int_0^a M(x, 0)\, dy \ . \end{aligned}$$

   By comparing these two formulas, we conclude

$$\int_C M dx + N dy = \iint_D (N_x - M_y)\, dA \ .$$

2. Let D be the parallelogram formed by the lines $x + y = 1, x + y = 3, y = 2x - 3, y = 2x + 2$. Evaluate the line integral

$$\oint_C dx + 3xy\, dy$$

   where C is the boundary of D oriented in anticlockwise direction. Suggestion: Try Green's theorem and then apply change of variables formula.

   Solution. By Green's theorem

$$\oint_C dx + 3xy\, dy = \iint_D 3y\, dA(x, y) \ .$$

Next, let $u = x + y$ and $v = y - 2x$. Then $(u, v) \mapsto (x, y)$ sends the rectangle $R = [1, 3] \times [-3, 2]$ to D. We have $\dfrac{\partial(u,v)}{\partial(x,y)} = 3$ and $x = (u - v)/3$ and $y = (2u + v)/3$. By the change of variables formula

$$\begin{aligned}\iint_D 3y dA(x,y) &= \iint_R (2u+v)\frac{1}{3}\, dA(u,v) \\ &= \frac{1}{3}\int_1^3 \int_{-3}^2 (2u+v)\, dvdu \\ &= \frac{1}{3}\int_1^3 (10u-5)\, du \\ &= \frac{35}{3}\ .\end{aligned}$$

3. Let $F = M\mathbf{i} + N\mathbf{j}$ be a smooth vector field in $\mathbb{R}^2$ except at the origin. Suppose that $M_y = N_x$. Show that for any simple closed curve γ enclosing the origin and oriented in anticlockwise direction, one has

$$\oint_\gamma M dx + N dy = \varepsilon \int_0^{2\pi} [-M(\varepsilon\cos\theta, \varepsilon\sin\theta)\sin\theta + N(\varepsilon\cos\theta, \varepsilon\sin\theta)\cos\theta]\, d\theta\ ,$$

for all sufficiently small ε. What happens when γ does not enclose the origin?

Solution. Let γ_ε be the circle entered at the origin with radius ε which is so small to be enclosed by γ. Then the vector field $\mathbf{F}$ is smooth in the region bounded by γ and γ_1. Applying Green's theorem in a multi-connected region we have

$$\oint_\gamma M dx + N dy = \oint_{\gamma'} M dx + N dy\ .$$

Using the standard parametrization, $\theta \mapsto (\varepsilon\cos\theta, \varepsilon\sin\theta)$, we further have

$$\oint_{\gamma'} M dx + N dy \ = \varepsilon \int_0^{2\pi} [-M(\varepsilon\cos\theta, \varepsilon\sin\theta)\sin\theta + N(\varepsilon\cos\theta, \varepsilon\sin\theta)\cos\theta]\, d\theta\ ,$$

for all sufficiently small ε.

The line integral vanishes when γ does not include the origin.